

Joint Channel Estimation and Nonlinear Distortion Compensation in OFDM Receivers

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Abstract—Nonlinear distortion in power amplifiers (PA) can significantly degrade performance of orthogonal frequency division multiplexed (OFDM) communication systems. This paper presents a joint maximum-likelihood channel frequency response and nonlinear PA model estimator for OFDM signals. Derivation of the estimator is based on Taylor-series representation of power amplifier nonlinearity and is suitable for wide range of memoryless PA models. A sub-optimal decision-aided algorithm for adaptive compensation of nonlinear distortion effects at the receiver-side is also presented. It is shown that the proposed algorithms can be used in IEEE 802.11a/g/p/ac compliant wireless LAN receivers without any modifications at the transmitter side. The performance of the proposed algorithms is studied by means of computer simulation.

Index Terms—OFDM, nonlinear distortion, power amplifier, channel estimation, iterative processing

I. INTRODUCTION

IN RECENT YEARS, orthogonal frequency-division multiplexing (OFDM) has emerged as a preferred candidate for a wide variety of wireless communication applications. OFDM has been used in wireless local area networks [1], in vehicular communication systems [2], in European digital terrestrial video broadcasting [3], and is a major contender for 5th generation mobile networks [4].

OFDM has several advantages over single-carrier systems, including spectral effectiveness, robustness to multi-path propagation and efficient implementation based on fast Fourier transform (FFT). Despite several advantages OFDM has one major drawback - high sensitivity to nonlinear distortions caused by the use of power amplifiers (PAs) at the transmitter [5], [6], [7], [8], [9]. The nonlinearity of PA causes interference both inside and outside the OFDM signal bandwidth. The out-of-band interference affects adjacent frequency channels, whereas the in-band interference results in degradation of system bit error rate (BER). Often, out-of-band spectral regrowth is a more serious problem, but in high throughput wireless applications, BER degradation may become unacceptable even when out-of-band spectral regrowth is tolerable.

There are a number of methods that can be implemented in OFDM transmitters in order to reduce performance degradation caused by PA nonlinearity. These methods include deliberate clipping schemes [10], reduced peak-to-average power ratio (PAPR) coding [21], and amplifier pre-distortion techniques [22]. However, such techniques may be hard to implement, for example, in low-cost mobile terminals for vehicular communication systems or wireless local area network

applications due to power, complexity or cost constraints. In such a case, the receiver-side nonlinearity compensation may be an attractive alternative for uplink processing, where more computational resources are available at the base station.

Recent studies [13], [14] show that the nonlinear amplification of OFDM signals combined with maximum-likelihood decoding at the receiver may deliver better BER performance than that of linear OFDM transmission. Unfortunately, true maximum-likelihood decoding is too complex for practical implementation. Recently, there have been several studies devoted to the sub-optimal reconstruction of nonlinearly distorted OFDM signals at the receiver side [15], [16], [17], [18], [19]. These techniques permit implementation of nonlinear OFDM decoders with intermediate complexity. Nonetheless, most studies assume that the receiver knows the transmitter nonlinear transfer function, and that a perfect channel state information is available at the receiver. These assumptions are somewhat artificial, therefore previous research has tended to focus on compensation of special types of nonlinearity, such as deliberate clipping for PAPR reduction [16], [17], [18], rather than on realistic distortions introduced in PA. To the best of author knowledge the problem of joint channel response and PA model estimation has not been addressed in the previous studies. In [19], the authors propose the receiver-side nonlinearity compensation with adaptive PA model estimation, without assuming perfect knowledge of channel state information at the receiver. However, the authors in [19], rely on conventional channel estimation techniques developed for linear multipath channels, and therefore either require special low-PAPR training symbols to minimize nonlinear distortion effect on channel estimation or incur performance penalty due to imperfect channel estimation.

In this paper, we propose an adaptive channel estimation and nonlinear distortion compensation algorithm for OFDM receivers. The proposed algorithm does not assume a perfect channel state information and prior knowledge of PA nonlinear transfer function at the receiver, and can be used to jointly estimate and compensate the channel response and PA nonlinearity using regular OFDM signal structure with block type pilot arrangements, and also mitigate the nonlinear distortion effects in decision-directed mode. The major difference between our approach and previous studies is that we rely on frequency-domain representation of PA nonlinearity. It simplifies derivation of joint maximum likelihood channel and PA model estimator and permits nonlinear distortion compensation solely in a frequency domain without costly conversions from frequency-domain to time-domain representation on every algorithm iteration.

The paper is organized as follows. In Section II, the model of OFDM system with nonlinear PA is defined. In Section III, a joint maximum likelihood channel and PA model estimator and a sub-optimal iterative decision-directed algorithm for detection of nonlinearly amplified OFDM signals are presented and performance of the proposed algorithms is studied by means of simulation in Section IV. Finally, Section V draws conclusions.

II. SYSTEM MODEL

Let us first introduce the OFDM transmission system shown in fig. 1. In the OFDM transmitter, information bits are mapped into baseband symbols $\{S_k\}$ using m -ary phase-shift-keying (PSK) or quadrature-amplitude-modulation (QAM) format. During active symbol interval a block of N complex baseband symbols $\mathbf{S} = [S_0, S_1, \dots, S_{N-1}]$ (possibly encoded by forward error correcting code) is transformed by means of inverse discrete Fourier transform (IDFT) and digital-to-analog conversion to the baseband OFDM signal as

$$z(t) = \sum_{k=0}^{N-1} S_k e^{j2\pi k \Delta f t}, \quad 0 < t < T_s, \quad (1)$$

where N is the number of sub-carriers, Δf is the separation between adjacent sub-carriers, and T_s is the active symbol interval. In practical OFDM systems, a cyclic prefix is usually added to every symbol $z(t)$. The cyclic prefix is a periodic extension of the symbol $z(t)$, which is primarily used to simplify equalizer design at the receiver side. We also assume that an OFDM signal contains a block of pilot symbols or set of pilot subcarriers to facilitate channel estimation and carrier and timing recovery at the receiver.

Two types of PA are mostly used in modern communication systems: travelling waves tube amplifiers (TWTA) and solid-state power amplifiers (SSPA). TWTA are used in high power satellite links while SSPA are used in many other applications because of its small size. In this paper, we mainly focus our attention on transmitters with memory-less nonlinearity. Under such assumption the baseband signal distorted in a nonlinear PA can be expressed as

$$y(t) = F_A[|z(t)|] e^{j(\arg[z(t)] + F_P[|z(t)|])}, \quad (2)$$

where $F_A[x]$ and $F_P[x]$ are the AM/AM and AM/PM functions (AM/AM nonlinearity causes amplitude distortions which depend on amplitude of the signal, while AM/PM nonlinearity causes phase distortions which depend on amplitude of the signal). AM/AM and AM/PM nonlinearity for SSPA is well represented by Rapp's model [20]:

$$F_A[\rho] = \rho \left[1 + \left(\frac{\rho}{A_{sat}} \right)^v \right]^{-1/v}, \quad F_P[\rho] = 0, \quad (3)$$

where A_{sat} is the output saturation voltage and v is the smoothness factor. A typical value for v is 2 - 4.

Saleh [21] and Ghorbani [22] models are the two alternative representations of PA nonlinear transfer function suitable for description of TWTA and SSPA, respectively.

These models have gained a lot of popularity because they represent the PA nonlinearity by simple analytical expressions.

Unfortunately, these traditional models can only be applied to a narrow class of PA, characterized by a regular shape of AM/AM and AM/PM functions. Instead, for more general solution, we rely on memoryless polynomial model (Taylor series expansion) to represent arbitrary PA nonlinearity [23], [24]:

$$y(t) = \sum_{p=3,5,\dots}^P \beta_p [z(t)]^{(p+1)/2} [z^*(t)]^{(p-1)/2}, \quad (4)$$

where P is the highest order of nonlinearity, and $\{\beta_p\}$ is the baseband power-series coefficient.

It is shown [25] that the complex power-series (4) represents a general nonlinear transfer function and its odd-order coefficients can be extracted from AM/AM and AM/PM measurements. It is important to note that only the odd-order terms produce in-band distortions. The even-order distortions are filtered out in zonal filter and do not influence system bit or packet error rate.

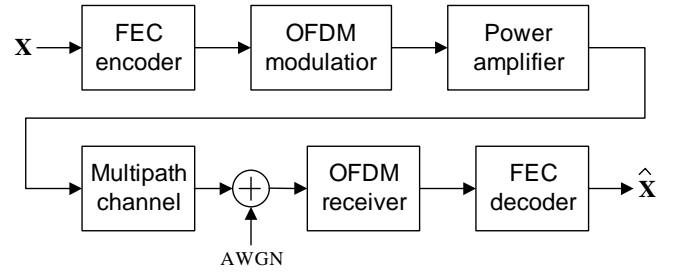


Figure 1. System model

III. PROPOSED ALGORITHMS

A. Effect of nonlinear PA on transmitted OFDM symbols

Let us consider the baseband OFDM signal at the output of nonlinear PA modeled by (4). Substituting (1) into (4) we immediately obtain

$$\begin{aligned} y(t) = & \beta_1 \sum_{k=0}^{N-1} S_k e^{j2\pi k \Delta f t} \\ & + \beta_3 \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} \sum_{n_3=0}^{N-1} S_{n_1} S_{n_2} S_{n_3}^* e^{j2\pi(n_1+n_2-n_3)\Delta f t} \\ & + \dots \end{aligned} \quad (5)$$

From (5) one can easily conclude that the OFDM transmitter with memoryless PA (4) is equivalent to a linear OFDM transmitter that emits modified baseband symbols S'_k , $k = 0, 1, \dots, N-1$:

$$\begin{aligned} S'_k = & \beta_1 S_k + \beta_3 \times \sum_{n_1+n_2-n_3=k} S_{n_1} S_{n_2} S_{n_3}^* \\ & + \beta_5 \times \sum_{\substack{n_1+n_2+n_3 \\ -n_4-n_5=k}} S_{n_1} S_{n_2} S_{n_3} S_{n_4}^* S_{n_5}^* + \dots \end{aligned} \quad (6)$$

Note that $S'_k \neq 0$ for $k < 0$ and $k > N-1$, which results in out-of-band emission. However, we do not consider the out-of-band components and assume that these are filtered out in transmitter and/or receiver filters.

It can be noted that the OFDM sub-carriers after nonlinear transformation are no longer orthogonal; therefore the optimal maximum-likelihood (ML) receiver requires a joint detection of transmitted vector $\mathbf{S} = [S_0, S_1, \dots, S_{N-1}]$ (see [13], [14] for detailed discussion). Unfortunately, the optimal ML solution has little practical value. It cannot be used in OFDM systems with large and intermediate number of sub-carriers due to extremely high complexity.

Providing that N is sufficiently large, a frequency-domain OFDM symbol distorted in nonlinear PA (6) can be represented on the basis of extended Bussgang theorem [5], [6], [26] as

$$\mathbf{S}' = \alpha \mathbf{S} + \mathbf{d}, \quad (7)$$

where $\mathbf{S}' = [S'_0, S'_1, \dots, S'_{N-1}]$, α is the complex attenuation factor, and $\mathbf{d} = [d_0, d_1, \dots, d_{N-1}]$ is the uncorrelated nonlinear distortion term.

After transformation in multipath channel and conventional DFT-based demodulation the received signal vector $\mathbf{R} = [R_0, R_1, \dots, R_{N-1}]$ can be expressed as

$$\mathbf{R} = \Theta \mathbf{S}' + \mathbf{w} = \alpha \Theta \mathbf{S} + \Theta \mathbf{d} + \mathbf{w}, \quad (8)$$

where $\mathbf{w} = [w_0, w_1, \dots, w_{N-1}]$ is the complex white Gaussian noise vector with i.i.d. components having zero-mean and variance σ_w^2 , and $\Theta = \text{diag}([H_0, H_1, \dots, H_{N-1}])$ is the diagonal matrix containing the frequency domain channel coefficients (i.e. discrete Fourier transform of the channel impulse response).

Consider the additive distortion term \mathbf{d} . From (7) it is straightforward to express \mathbf{d} as

$$\mathbf{d} = \mathbf{S}' - \alpha \mathbf{S}. \quad (9)$$

Now, using (6) we can represent \mathbf{S}' in a compact form

$$\mathbf{S}' = \beta_1 \mathbf{S} + \sum_{p=3,5,\dots}^P \beta_p \mathbf{d}^{(p)}, \quad (10)$$

where elements of $\mathbf{d}^{(p)} = [d_0^{(p)}, d_1^{(p)}, \dots, d_{N-1}^{(p)}]$ can be computed by means of the discrete convolution:

$$\begin{aligned} \{d_k^{(3)}\} &= \{S_k\} * \{S_k\} * \{S_{N-k}^*\}, \\ \{d_k^{(5)}\} &= \{S_k\} * \{S_k\} * \{S_k\} * \{S_{N-k}^*\} * \{S_{N-k}^*\}, \\ &\dots \\ \{d_k^{(P)}\} &= \underbrace{\{S_k\} * \{S_k\} * \dots * \{S_{N-k}^*\}}_{P\text{-sequences}}, \end{aligned} \quad (11)$$

and $\{S_{N-k}^*\}$ represents the complex conjugated and reversely ordered sequence of $\{S_k\}$. Note that $\mathbf{d}^{(p)}$ can efficiently be computed by zero-padding sequences $\{S_k\}$ and $\{S_{N-k}^*\}$, taking IDFT, multiplying appropriate coefficients of IDFT, taking DFT and removing out-of-band components [27]. It is also

worth noting that computational complexity of convolutions (11) can further be reduced by taking into account a finite resolution of $\{S_k\}$ (typically 1-4 bit).

After simple manipulations, it is straightforward to obtain

$$\mathbf{d} = \alpha \left[(c_1 - 1) \mathbf{S} + \sum_{p=3,5,\dots}^P c_p \mathbf{d}^{(p)} \right], \quad (12)$$

where $c_p = \beta_p \alpha^{-1}$ is the p -th normalized coefficient of PA transfer function. As one can see from (12) the distortion term is a function of transmitted symbol $\mathbf{S} = [S_0, S_1, \dots, S_{N-1}]$, and $(P+1)/2$ unknown coefficients c_1, c_3, \dots, c_P . Conversely, the complex PA gain α can be expressed using baseband power-series coefficients $\{\beta_p\}$ as

$$\alpha = \beta_1 + \sum_{p=3,5,\dots}^P \beta_p T^{(p)}, \quad (13)$$

where term $T^{(p)}$ represents an average energy of all p -th order distortion terms that produce scaled replica of information-bearing signal \mathbf{S} . Generally, value of $T^{(p)}$ depends on the number of subcarriers and the modulation scheme, and can be evaluated either analytically, or numerically. For constant amplitude modulation schemes (such as m -ary PSK), $T^{(p)}$ coincides with the number of all p -th order distortion terms that produce scaled replica of \mathbf{S} (see [8], for $p = 3, 5, 7, 9$ and $p \rightarrow \infty$). Derivation of $T^{(p)}$ for $p = 3, 5$ and non-constant amplitude modulation (e.g. m -ary QAM) is given in Appendix A.

Dividing both sides of (13) by α , we can express coefficient c_1 as

$$c_1 = 1 - \sum_{p=3,5,\dots}^P c_p T^{(p)}, \quad (14)$$

and by substituting (14) in (12) the distortion term can be expressed as

$$\mathbf{d} = \alpha \sum_{p=3,5,\dots}^P c_p \left(\mathbf{d}^{(p)} - T^{(p)} \mathbf{S} \right). \quad (15)$$

Finally, the frequency-domain received signal can be reformulated using (15) as:

$$\mathbf{R} = \Theta_\alpha \left(\mathbf{S} + \sum_{p=3,5,\dots}^P c_p \left(\mathbf{d}^{(p)} - T^{(p)} \mathbf{S} \right) \right) + \mathbf{w}, \quad (16)$$

where $\Theta_\alpha = \text{diag}([\alpha H_0, \alpha H_1, \dots, \alpha H_{N-1}])$.

B. Joint maximum-likelihood estimation of channel response and normalized power-series coefficients

To simplify further derivations we assume that the time-domain channel impulse response (CIR) is not limited by cyclic prefix duration and, therefore, we do not take into account correlation of elements in vector $\mathbf{H} = [H_0, H_1, \dots, H_{N-1}]$.

We aim to find a joint maximum likelihood estimator of vectors $\mathbf{H}' = [\alpha H_0, \alpha H_1, \dots, \alpha H_{N-1}]$ and $\mathbf{c} = [c_3, c_5, \dots, c_P]$, given a group of M received OFDM symbols $\mathbf{R}^{(m)} = [R_0^{(m)}, R_1^{(m)}, \dots, R_{N-1}^{(m)}]$, $m = 1, 2, \dots, M$. Since the noise term \mathbf{w} in (16) is i.i.d Gaussian distributed, the maximum-likelihood estimator is equivalent to least-squares estimator that minimizes the error function [28]:

$$J = \sum_{m=1}^M \left\| \mathbf{R}^{(m)} - \Theta_{\alpha} \left(\mathbf{S}^{(m)} + \mathbf{U}^{(m)} \mathbf{c} \right) \right\|^2 \quad (17)$$

where

$$\mathbf{U}^{(m)} = \mathbf{d}^{(p,m)} - T^{(p)} \mathbf{S}^{(m)} \quad (18)$$

Condition (17) leads to the system of $M \times N$ nonlinear equations for $N + (P - 1)/2$ unknown parameters:

$$R_k^{(m)} - H'_k \left(S_k^{(m)} + \sum_{p=3,5,\dots}^P c_p U^{(p,m)} \right), \quad (19)$$

$$k = 0, 1, \dots, N - 1$$

$$m = 1, 2, \dots, M$$

where $U^{(p,m)} = d_k^{(p,m)} - T^{(p)} S_k^{(m)}$. Note that at least two *different* OFDM symbols are required to jointly estimate channel frequency response and normalized power-series coefficients since the total number of equations in (19) should be no less than $N + (P - 1)/2$.

Unfortunately, the solution of (19) cannot be expressed in a simple closed-form. Nonetheless, sub-optimal iterative methods can be used to solve (19). Here, we propose one such method.

Firstly, we note that the channel response vector \mathbf{H}' that minimizes J for given \mathbf{c} can be found as:

$$\hat{\mathbf{H}}'_k = \frac{1}{M} \sum_{m=1}^M R_k^{(m)} \left(S_k^{(m)} + \sum_{p=3,5,\dots}^P c_p U^{(p,m)} \right)^{-1}, \quad (20)$$

$$k = 0, 1, \dots, N - 1$$

One can easily observe that equation (20) is equivalent to a conventional least-squares channel estimation, albeit using the modified reference symbols $S_k^{(m)} + \sum_{p=3,5,\dots}^P c_p \left(d_k^{(p,m)} - T^{(p)} S_k^{(m)} \right)$ and averaged over M OFDM training symbols. On the other hand, the power-series coefficient vector \mathbf{c} that minimizes J for given \mathbf{H}' can be found using linear least-squares solution:

$$\hat{\mathbf{c}} = \left(\mathbf{U}^H \mathbf{\Lambda}^H \mathbf{\Lambda} \mathbf{U} \right)^{-1} \mathbf{U}^H \mathbf{\Lambda}^H (\mathbf{R} - \mathbf{\Lambda} \mathbf{S}) \quad (21)$$

where $\mathbf{U} = [\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \dots, \mathbf{U}^{(M)}]^T$, $\mathbf{R} = [\mathbf{R}^{(1)}, \mathbf{R}^{(2)}, \dots, \mathbf{R}^{(M)}]^T$, $\mathbf{S} = [\mathbf{S}^{(1)}, \mathbf{S}^{(2)}, \dots, \mathbf{S}^{(M)}]^T$, and $\mathbf{\Lambda} = \text{diag}[H'_0, H'_1, \dots, H'_{N-1}, H'_0, \dots, H'_{N-2}, H'_{N-1}]$.

Then the solution of (19) can be approximated using the following iterative algorithm:

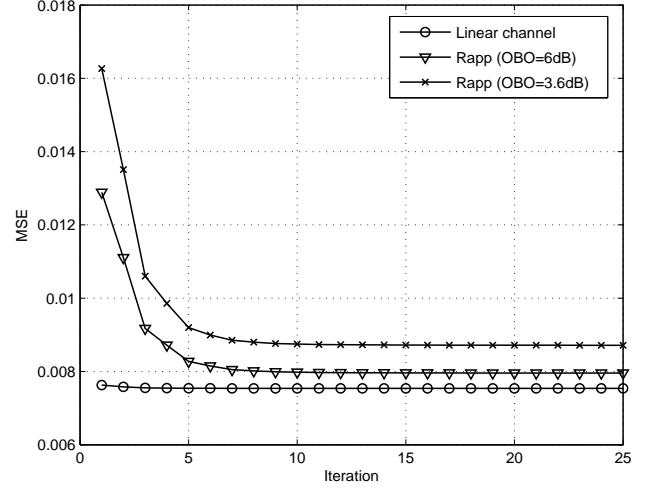


Figure 2. Convergence of joint ML channel and PA model estimator (Rapp PA model, $v=2$, $N=120$, $M=2$, $\text{SNR}=20$ dB)

Algorithm 1 Joint estimation of channel response and PA model

- 1) Set the initial estimate $\hat{\mathbf{c}} = 0$
 - 2) Calculate the channel frequency response estimate $\hat{\mathbf{H}}'$ in accordance with (20) using current estimate of $\hat{\mathbf{c}}$
 - 3) Calculate the estimate of power-series coefficients \mathbf{c} in accordance with (21) using current estimate of $\hat{\mathbf{H}}'$
 - 4) Repeat steps 2) and 3) until the error function J is no longer decreasing or the maximum number of iterations is reached
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Although there is no guarantee that the proposed iterative procedure will always converge to the global minimum, our numerical simulations indicate that in most practical scenarios (i.e. typical noise levels, multipath channel models, and amplifier nonlinearity) the proposed iterative algorithm quickly converges to a steady-state solution that minimizes (17). Learning curves obtained via simulation for a system with two training symbols at different values of output back-off (OBO) are illustrated in figure 2. In most cases, the proposed procedure converges to a steady-state value of J within 4-5 full iterations.

The MSE gain that can be achieved by using the proposed joint channel and PA model estimator over a conventional least-squares channel estimator in nonlinear channel (Rapp PA model) is illustrated in fig. 3. Not surprisingly, the improvement in terms of MSE is higher in low OBO and high SNR region where the MSE of the conventional least-squares channel estimator is dominated by nonlinear distortion noise.

It should be noted that the proposed joint maximum-likelihood estimation of channel frequency response and normalized power-series coefficients has moderate complexity. If the pilot symbols \mathbf{S} are known *a priori*, the distortion vectors $\mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \dots, \mathbf{U}^{(M)}$, can be pre-computed. The channel frequency response calculation step (20) requires only N complex divisions, and since P is usually relatively small, the calculation of (21) is also fairly simple. For example, for

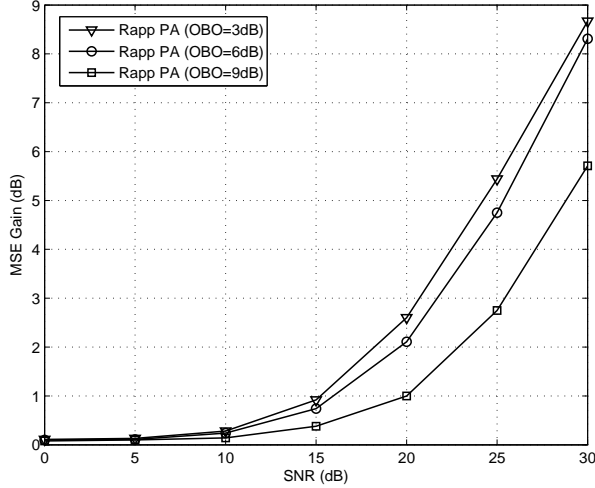


Figure 3. MSE gain of the proposed joint channel and PA model estimator over a conventional least-squares channel estimator (Rapp PA model, $v=2$, $N=120$, $M=2$)

$P = 5$, the term $\mathbf{U}^H \mathbf{\Lambda}^H \mathbf{\Lambda} \mathbf{U}$ in (21) is a 2×2 matrix. Furthermore, since we estimate both channel frequency response and power amplifier coefficients in frequency domain there are no costly DFT/IDFT operations at every iteration of the algorithm.

C. Decision-aided nonlinear distortion compensation

After applying a conventional zero-forcing equalization to the received signal (16) the equalizer output (without Gaussian noise term for simplicity) can be expressed as

$$R_k^{(eq)} = \frac{R_k}{H_k^i} = S_k + \sum_{p=3,5,\dots}^P c_p \left(\hat{d}_k^{(p)} - T^{(p)} S_k \right) \quad (22)$$

The distortion term in (22) can be canceled using an iterative decision-aided approach [15]. At the first step, the equalized signal (22) is utilized to obtain tentative decisions $\hat{\mathbf{S}}$, e.g. using slicer or a feedback from FEC decoder. Then, we can reconstruct distortion term $\sum_{p=3,5,\dots}^P \hat{c}_p \left(\hat{d}_k^{(p)} - T^{(p)} \hat{S}_k \right)$ and subtract it from the output of zero-forcing equalizer (22). Since the tentative decisions may contain errors the reconstructed distortion term might be inaccurate. However, if the number of errors in $\hat{\mathbf{S}}$ is relatively small the compensated signal will be less noisy than the original equalizer output and the decisions obtained at the second iteration will contain fewer errors. The proposed decision-aided nonlinearity compensation algorithm can be summarized as follows:

Algorithm 2 Iterative decision-aided nonlinearity mitigation

- 1) Use equalized signal (22) to obtain tentative decisions $\hat{\mathbf{S}}$ (via slicer of FEC decoder)
- 2) Calculate $\hat{\mathbf{d}}^{(p)} = [\hat{d}_0^{(p)}, \hat{d}_1^{(p)}, \dots, \hat{d}_{N-1}^{(p)}]$, $p = 3, 5, \dots, P$.
- 3) Compensate the nonlinear distortion term $R_k^{(comp)} = R_k^{(eq)} - \sum_{p=3,5,\dots}^P \hat{c}_p \left(\hat{d}_k^{(p)} - T^{(p)} \hat{S}_k \right)$, $k = 0, 1, \dots, N-1$
- 4) Use distortion compensated signal $R_k^{(comp)}$ to obtain updated decisions $\hat{\mathbf{S}}$
- 5) Repeat steps 2)-4) until $\hat{\mathbf{S}}$ is no longer differs from $\hat{\mathbf{S}}$ obtained at previous iteration or until the maximum number of iterations is reached.

In addition to distortion compensation, the decision-aided approach can be also used to update estimate of $\hat{\mathbf{c}}$. In particular, it is possible to update non-linear model parameters at each iteration of the algorithm using (21) with $M=1$ and tentative decision vector $\hat{\mathbf{S}}$. However, due to possibility of errors in tentative decisions $\hat{\mathbf{S}}$ it is preferable to average these estimates over several OFDM symbols to reduce the effect of decision errors in a single OFDM symbol. One simple way to achieve this it is to employ the first order recursive filter

$$\hat{\mathbf{c}}_t^{(avg)} = \gamma \cdot \hat{\mathbf{c}}_t + (1 - \gamma) \cdot \hat{\mathbf{c}}_{t-1}^{(avg)}, \quad (23)$$

where where t is the OFDM symbol index, and γ is the smoothing factor. The optimal choice of γ shall be discussed in the next section.

D. Limitations

The proposed channel estimation and nonlinear distortion compensation technique has a few limitations:

- Firstly, the Busgang theorem is only applicable if the number of OFDM sub-carriers $N \rightarrow \infty$. Yet, many state-of-the-art wireless communication schemes employ OFDM modulation with relatively small number of sub-carriers (e.g. in IEEE802.11a/g/p, the number of active subcarriers is $N=52$). From (32) one can easily see that for non-constant amplitude modulation (such as m -ary QAM) and finite number of OFDM subcarriers the scaling factor α in (7) is not a constant value as suggested by Busgang theory, but varies from symbol-to-symbol and from subcarrier-to-subcarrier.
- Secondly, the polynomial model parameters $\hat{\mathbf{c}}$ are fitted to the training data that may have limited dynamic range, which in turn may result in model underfitting. In particular, many packet-based OFDM systems rely on training symbols with reduced PAPR to decrease channel estimation noise caused by nonlinearities in conventional channel estimators. Such arrangement, although beneficial for conventional channel estimation schemes, limits the efficiency of the proposed joint channel and non-linear model estimation algorithm, because the nonlinear PA model may be incorrectly reconstructed outside of the dynamic range represented by training symbols.
- Thirdly, since the proposed nonlinear distortion compensation technique relies on decision feedback mechanism,

it is susceptible to error propagation effects. In fact, if the first tentative decision vector $\hat{\mathbf{S}}$ contains too many errors the reconstructed distortion term may significantly differ from a true distortion term and the SNR of the compensated signal $R_k^{(comp)}$ may become even worse than the SNR at the output of the conventional zero-forcing equalizer (22).

Several ad-hoc techniques can be employed to reduce the error propagation effect. For example, one can utilize decision feedback from FEC decoder. However, in such a case, caution should be used to avoid further deterioration of performance due to error correlation. For example, the bit errors produced by Viterbi decoding are correlated and tend to group together in error bursts. Therefore, if the number of bits transmitted per OFDM symbol is relatively small and there is no interleaving between OFDM symbols the hard-decision feedback from Viterbi decoder may significantly deteriorate performance of the proposed decision-aided nonlinear distortion compensation algorithm. Another simple method to partially reduce the error-propagation effect is to intentionally decrease the compensation term at first iterations of the decision-aided algorithm, i.e. $R_k^{(comp)} = R_k^{(eq)} - \mu_i \sum_{p=3,5,\dots}^P \hat{c}_p (\hat{d}_k^{(p)} - T^{(p)} \hat{S}_k)$, $k = 0, 1, \dots, N-1$, where $\mu_i \leq 1$ is the damping factor at i -th iteration, such that $\mu_i \leq \mu_{i+1}$. This technique has demonstrated some performance gain in OFDM systems with small number of subcarriers (see simulation section for details).

E. Application of the proposed technique to IEEE802.11ac compliant receivers

The proposed joint channel and non-linear model estimation and distortion compensation scheme can be used in standard-complaint IEEE802.11ac receivers with minor modifications. We consider a system with single transmit chain, because in practice it is very likely that the power amplifier nonlinearity will occur in low-cost mobile terminals that usually rely on a single transmitter due to power, size and cost constraints. The proposed channel and nonlinear model estimation technique requires, at least, two different OFDM training symbols for initial channel estimation. To satisfy this condition in IEEE802.11a/g/p or in legacy mode of IEEE802.11ac [1], one can use the L-LTF training symbol and the demodulated and reconstructed L-SIG symbol. Similarly, in very high throughput (VHT) mode, one can use the VHT-LTF and SIG-B symbols for initial channel and nonlinear model estimation. L-SIG and SIG-B symbols are not known *a priori* at the receiver-side, however, they employ very robust 1/2-rate convolutional coding and BPSK modulation [1], and therefore, can be demodulated and reconstructed without errors in all channel conditions that may be deemed suitable for 64-QAM and 256-QAM modes. After the initial channel and nonlinear model estimation the nonlinear model parameters \hat{c} can be constantly updated in decision-directed way as described in the previous section.

IV. SIMULATION RESULTS AND DISCUSSION

To evaluate performance of the proposed scheme we simulated IEEE802.11ac compliant single-input single-output

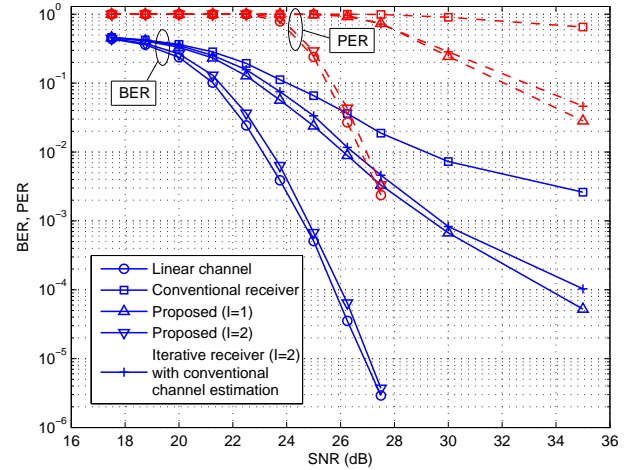


Figure 4. BER/PER vs SNR for IEEE802.11ac compliant receiver ($N=484$, 256-QAM, code rate 3/4) in AWGN channel, Rapp PA ($v=2$, OBO=9.7dB)

(SISO) transceiver in VHT mode and in legacy mode (thus, the results are also applicable to IEEE802.11a/g/p). Perfect carrier frequency offset and symbol timing synchronization is assumed. We primarily focus on 64-QAM and 256-QAM modes, because in these modes the power amplifier nonlinearity may cause significant bit and packet error rate degradation even though the spectral regrowth may be well within spectral mask requirements [1, p.298]. In all simulations, we assume that the channel and PA model estimator is initialized at every transmitted burst. PA nonlinearity was modeled by Rapp model (3) with parameter $v = 2$. For fair comparison, the linear receiver also relies on two OFDM symbols L-LTF (or VHT-LTF) and the reconstructed L-SIG (or SIG-B) for least-squares channel estimation. We simulated the receiver performance in AWGN and block fading multipath channels with delay profile models proposed in [29] for typical WLAN indoor environment. The packet size in all simulation cases was set to 400 bytes.

Simulation results for VHT mode (160MHz bandwidth and 484 active subcarriers) are presented in figures 4 and 5. As one can see the proposed algorithms allow almost perfect compensation of mild nonlinear distortions in IEEE802.11ac VHT mode both in AWGN and typical multipath channels. The proposed receiver performance is just a fraction of dBs away from linear transmission after two or three iterations. It should be noted that the proposed joint channel and PA model estimation plays a key role in the overall performance improvement. In particular, the decision-aided iterative nonlinearity compensation algorithm combined with a conventional channel estimation scheme demonstrates substantially worse performance than the proposed receiver with joint channel and PA model estimation (fig. 4, 5).

Simulation results for legacy mode (20MHz bandwidth and 52 active subcarriers) are presented in fig. 6 and 7. In legacy mode, the performance improvement is also significant, although the overall performance is slightly worse than that in VHT mode. In addition, to get the best results it was

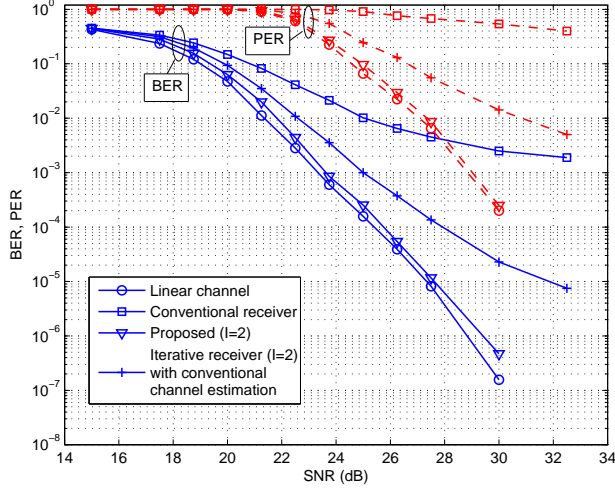


Figure 5. BER/PER vs SNR for IEEE802.11ac compliant receiver ($N=484$, 64-QAM, code rate 3/4) in block fading multipath channel (Model B [29]), Rapp PA ($v=2$, OBO=8.6dB)

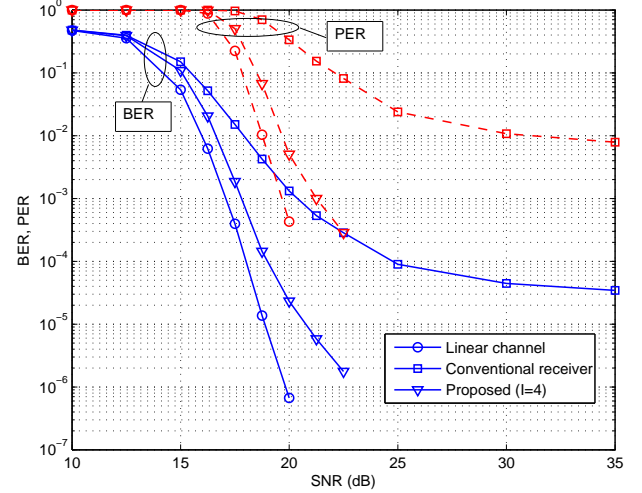


Figure 6. BER/PER vs SNR for IEEE802.11ac compliant receiver in legacy mode ($N=52$, 64-QAM, code rate 3/4), AWGN, Rapp PA ($v=2$, OBO=8.5dB)

necessary to increase the number of iterations and downscale the compensation term at the first iteration ($\mu_1 = 0.75$) to minimize the adverse effect of error propagation. The main reason for slightly worse performance in legacy mode is the lower number of active subcarriers per OFDM symbol. The effect of reducing the number of active subcarriers is two-fold. Firstly, a fewer data bits per OFDM symbol means that accidental tentative decision errors at the first step of decision-aided compensation algorithm may have larger impact on algorithm convergence. Secondly, as we discussed earlier, in case of small number of OFDM subcarriers, the Busgang theorem is no longer valid, and as a result the scaling factor α in (7) is no longer constant, but varies from OFDM symbol to OFDM symbol. It means that the optimal set of coefficients $\hat{\mathbf{c}}$ that minimizes distortion function (17) for training OFDM symbols may be highly sub-optimal for some OFDM data symbols. This fact is illustrated in figure 8, where we plot simulation results for legacy mode ($N=52$) using ideal decision feedback instead of slicer decisions. One may expect that the ideal decision feedback should always result in a perfect compensation of nonlinear distortions, however, this is not the case. If estimation of $\hat{\mathbf{c}}$ is averaged over all OFDM symbols in a packet, the performance curves exhibit error-floor behavior at relatively high BER/PER. The issue can be solved by re-estimating $\hat{\mathbf{c}}$ at every OFDM symbol independently, which is equivalent to setting $\gamma = 1$ in (23). However, this approach demonstrates poor performance with non-ideal decision feedback, since relying on incorrect decisions $\hat{\mathbf{S}}$ may significantly affect estimation of parameter vector $\hat{\mathbf{c}}$ and may lead to considerable performance degradation. Therefore, the optimal smoothing factor γ was found experimentally; in most cases, it was found to be in the range of $\gamma = 0.2 \div 0.3$.

It should be noted that the proposed algorithm is based on memoryless model of power amplifier. This approach is mainly justified by the fact that the low-power amplifiers used in mobile terminals usually exhibit very weak memory

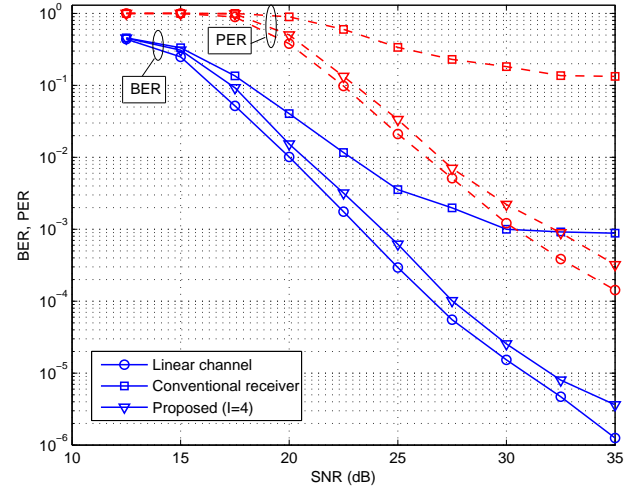


Figure 7. BER/PER vs SNR for IEEE802.11ac compliant receiver in legacy mode ($N=52$, 64-QAM, code rate 3/4) in block fading multipath channel (Model A [29]), Rapp PA ($v=2$, OBO=8.5dB)

effects. However, the proposed technique can be also applied to the compensation of nonlinear distortion effects caused by PA with memory. In particular, a combination of memoryless PA and frequency-selective fading channel can be viewed as a Hammerstein nonlinear model, which is often used to model memory effects in PAs and demonstrates good modeling behavior [30]. Moreover, our simulation results (not shown here for brevity) indicate that the proposed joint channel estimation and distortion compensation technique provides substantial performance gain for Wiener-type PA models.

V. CONCLUSIONS

Novel algorithms for joint channel response and PA model estimation and decision-aided nonlinear distortion compensation for nonlinearly distorted OFDM signals have been

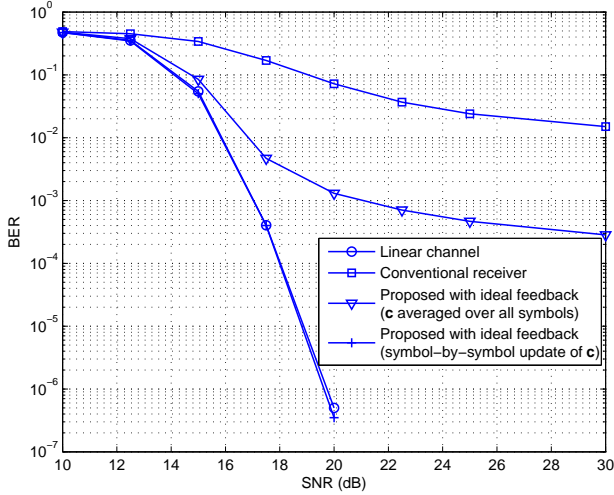


Figure 8. BER vs SNR for IEEE802.11ac compliant receiver in legacy mode ($N=52$, 64-QAM, code rate 3/4) in AWGN, Rapp PA ($v=2$, OBO=6dB) with ideal decision feedback

proposed. The proposed algorithms are suitable for a typical OFDM communication system with block type pilot arrangement. In particular, they can be used in standard-compliant IEEE802.11ac receivers. The proposed receiver performance has been evaluated in terms of bit and packet error rate through computer simulations, demonstrating the significant benefits of the proposed algorithms over conventional linear receivers, especially in high throughput modes of operation.

APPENDIX A

COMPLEX ATTENUATION FACTOR AND UNCORRELATED NONLINEAR DISTORTION TERM ($p=5$)

Consider k -th subcarrier of OFDM symbol transformed in nonlinear PA modeled by fifth-order polynomial model. Its representation is given by (6). It can be observed that the third- and fifth-order terms on the right-hand side of equation (6) produce the scaled replica of transmitted symbol S_k if one of the following conditions is met:

- 1) For the third-order terms:

$$A_1^{(3)} : ([n_1 = k] \& [n_2 = n_3]) \text{ or}$$

$$A_2^{(3)} : ([n_2 = k] \& [n_1 = n_3])$$

- 2) For the fifth-order terms:

$$A_1^{(5)} : ([n_1 = k] \& [n_2 = n_4] \& [n_3 = n_5]) \text{ or}$$

$$A_2^{(5)} : ([n_1 = k] \& [n_2 = n_5] \& [n_3 = n_4]) \text{ or}$$

$$A_3^{(5)} : ([n_2 = k] \& [n_1 = n_4] \& [n_3 = n_5]) \text{ or}$$

$$A_4^{(5)} : ([n_2 = k] \& [n_1 = n_5] \& [n_3 = n_4]) \text{ or}$$

$$A_5^{(5)} : ([n_3 = k] \& [n_1 = n_4] \& [n_2 = n_5]) \text{ or}$$

$$A_6^{(5)} : ([n_3 = k] \& [n_1 = n_5] \& [n_2 = n_4]).$$

The sum of all third-order terms for which at least one of conditions $A_1^{(3)}$ or $A_2^{(3)}$ is met can be calculated using the equality from combinatorial theory [31]:

$$|A_1^{(3)} \cup A_2^{(3)}| = |A_1^{(3)}| + |A_2^{(3)}| - |A_1^{(3)} \cap A_2^{(3)}| \quad (24)$$

where $|\cdot|$ denotes the sum of all terms for which given condition is met. If for given set of indexes k, n_1, n_2, n_3 condition $A_1^{(3)}$ is met the third-order term becomes

$$|A_1^{(3)}| = \beta_3 S_k \sum_{t=0}^{N-1} |S_t|^2 \quad (25)$$

The same result can be obtained for condition $A_2^{(3)}$. It is easy to show that

$$|A_2^{(3)}| = |A_1^{(3)}| \quad (26)$$

It can be noted that both conditions $A_1^{(3)}$ and $A_2^{(3)}$ are met simultaneously if $n_1 = n_2 = n_3 = k$. For given k there is only one third-order term, which satisfies condition $n_1 = n_2 = n_3 = k$. Thus, $|A_1^{(3)} \cap A_2^{(3)}|$ is determined by

$$|A_1^{(3)} \cap A_2^{(3)}| = \beta_3 S_k |S_k|^2 \quad (27)$$

Finally, the sum of all third-order terms, which produce scaled replica of S_k , can be expressed as

$$|A_1^{(3)} \cup A_2^{(3)}| = \beta_3 S_k \left(2 \sum_{t=0}^{N-1} |S_t|^2 - |S_k|^2 \right) \quad (28)$$

Similarly, the sum of all fifth-order terms for which at least one of conditions $A_i^{(5)}, i = 1, 2, \dots, 6$ is met can be calculated using the following equality [31]:

$$\begin{aligned} & |A_1^{(5)} \cup A_2^{(5)} \cup A_3^{(5)} \cup A_4^{(5)} \cup A_5^{(5)} \cup A_6^{(5)}| \\ &= \sum_{i=1}^6 |A_i^{(5)}| - \sum_{i < j} |A_i^{(5)} \cap A_j^{(5)}| \\ &+ \sum_{i < j < k} |A_i^{(5)} \cap A_j^{(5)} \cap A_k^{(5)}| \\ &- \sum_{i < j < k < t} |A_i^{(5)} \cap A_j^{(5)} \cap A_k^{(5)} \cap A_t^{(5)}| \\ &+ \sum_{i < j < k < t < u} |A_i^{(5)} \cap A_j^{(5)} \cap A_k^{(5)} \cap A_t^{(5)} \cap A_u^{(5)}| \\ &- |A_1^{(5)} \cap A_2^{(5)} \cap A_3^{(5)} \cap A_4^{(5)} \cap A_5^{(5)} \cap A_6^{(5)}| \end{aligned} \quad (29)$$

Due to space limitation we present here only the final result of (31) calculation. It can be shown that

$$\begin{aligned} |A_1^{(5)} \cup \dots \cup A_6^{(5)}| &= \beta_5 S_k \times 6 \sum_{t=0}^{N-1} \sum_{l=0}^{N-1} |S_t|^2 |S_l|^2 \\ &- \beta_5 S_k \times 6 \sum_{t=0}^{N-1} |S_t|^2 |S_k|^2 \\ &- \beta_5 S_k \times 3 \sum_{t=0}^{N-1} |S_t|^4 \\ &+ \beta_5 S_k \times 4 |S_k|^4 \end{aligned} \quad (30)$$

Now, making use of (28) and (30) we can rewrite expression (6) as

$$S'_k = \alpha_k S_k + d_k \quad (31)$$

where d_k is the uncorrelated nonlinear distortion term, and α_k is the complex attenuation factor that can be expressed as

$$\alpha_k = \beta_1 + \beta_3 T_k^{(3)} + \beta_5 T_k^{(5)} \quad (32)$$

where

$$T_k^{(3)} = 2 \sum_{l=0}^{N-1} |S_l|^2 - |S_k|^2 \quad (33)$$

and

$$T_k^{(5)} = 6 \sum_{l=0}^{N-1} \sum_{t=0}^{N-1} |S_l|^2 |S_t|^2 - 6 \sum_{t=0}^{N-1} |S_t|^2 |S_k|^2 - 3 \sum_{t=0}^{N-1} |S_t|^4 + 4 |S_k|^4 \quad (34)$$

Finally, the uncorrelated distortion term d_k can be expressed by

$$d_k = \beta_3 \left[d_k^{(3)} - T_k^{(3)} S_k \right] + \beta_5 \left[d_k^{(5)} - T_k^{(5)} S_k \right] \quad (35)$$

where $d_k^{(3)}$ and $d_k^{(5)}$ are calculated by (11). It worth noting that in case of m -QAM signaling α_k depends on all transmitted symbols $\{S_i\}$, $i = 0, 1, \dots, N-1$. However, it can be observed that for $N \rightarrow \infty$, α_k becomes very close to its average value $\alpha_k \approx \alpha$. If all symbols are transmitted independently and with equal probability, we can obtain for very large N

$$\alpha \approx \beta_1 + \beta_3 T^{(3)} + \beta_5 T^{(5)} \quad (36)$$

where

$$T^{(3)} = (2N-1) E \left[|s_l|^2 \right], \quad (37)$$

$$T^{(5)} = 6N(N-1) \left(E \left[|s_l|^2 \right] \right)^2 - (3N-4) E \left[|s_l|^4 \right], \quad (38)$$

and $E[\cdot]$ denotes expectation. $E \left[|s_l|^2 \right]$ and $E \left[|s_l|^4 \right]$ constants depend on constellation type.

In case of m -QAM signaling with in-phase and quadrature components $I, Q = \{-(\sqrt{m}-1), -(\sqrt{m}-3), \dots, +(\sqrt{m}-3), +(\sqrt{m}-1)\}$ straightforward calculation gives

$$E \left[|s_l|^2 \right] = \frac{2}{3} (m-1) \quad (39)$$

$$E \left[|s_l|^4 \right] = \frac{4}{45} (m-1) (7m-13) \quad (40)$$

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